a) i) In the two zero potential energy regions, the Schrödinger equation requires the wave function to be sinusoidal since $E > U$. (The specific value of λ will be dealt with in part b.) In the plateau region the Schrödinger equation requires the wave function to have zero second derivative (i.e., to be linear) since $E = U$. These requirements are clearly satisfied.
ii) At $x = \pm 2L$, where there is an infinite discontinuity in U, the requirement is that $\psi = 0$. At $x = \pm L$ the requirement is that ψ and its first derivative wrt x be continuous. We find that $\psi(\pm 2L) = 0$ and that $\psi(\pm L) = A$ and $\frac{d\psi}{dx}(\pm L) = 0$ thus satisfying the boundary conditions.

iii) For lower values of E the wavelength would necessarily be longer so that the wave function would go through *less* than a quarter of a wavelength in the low potential energy regions. It would enter the plateau region sloping away from the axis. (Draw a picture!) As a result it would exponentially grow in that region (since $E < U$) and be sloping even *faster* away from the axis when it got to the other low potential region. Thus, there would be no way to match it up in slope with a wavefunction in that last region that goes to zero as it must at the final boundary. Thus, the given wavefunction is the ground state energy wavefunction.

b) [1] The wavelength in the low potential energy regions is clearly $\lambda = 4L$ so $p = \frac{h}{4L}$.

c) [1] $E_g = U + K = 0 + \frac{p^2}{2m} = \frac{h^2}{32mL^2} = U_o$

d) [4] The normalization condition is that

$$1 = \int_{-\infty}^{\infty} \psi^2(x)\,dx = 2\int_0^{2L} \psi^2(x)\,dx = 2\left[\int_0^L A^2 dx + \int_L^{2L} A^2 \sin^2\left(\frac{\pi x}{2L}\right)dx\right]$$

Using the fact that the average value of $\sin^2$ is one half over any interval that starts and end at maxima (i.e. 1s) or minima (0s) we find

$$1 = 2\left[A^2 L + \frac{A^2}{2}L\right] = 3A^2 L \qquad \Rightarrow A = \frac{1}{\sqrt{3L}}$$

e) i) Due to the symmetry of the wavefunction about $x = 0$, $\langle x \rangle = 0$

ii) $\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi^2(x)\,dx = 2\left[\int_0^L x^2 A^2 dx + \int_L^{2L} x^2 A^2 \sin^2\left(\frac{\pi x}{2L}\right)dx\right] = 2A^2\left[\frac{L^3}{3} + \frac{L^3}{6}\left(7 - \frac{18}{\pi^2}\right)\right]$

where the second integral was performed with the help of Mathematica. Thus, $\langle x^2 \rangle = L^2\left(1 - \frac{2}{\pi^2}\right)$

iii) Again due to the symmetry of the wavefunction and the fact that at any position the particle has equal probability if moving to the left or right with the associated momentum, $\langle p \rangle = 0$.

iv) $\langle U \rangle = \int_{-\infty}^{\infty} U(x)\psi^2(x)\,dx = 2\left(\int_0^L A^2 U_o dx + \int_L^{2L} 0\,dx\right) = 2A^2 U_o L = \frac{2}{3}U_o$

iv) Using the hint, $U_o = E_g = \langle E \rangle = \langle K + U \rangle = \langle K \rangle + \langle U \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{2}{3}U_o$.

Thus, $\langle p^2 \rangle = \frac{2}{3} m U_o = \frac{2}{3} m \left(\frac{h^2}{32mL^2} \right) = \frac{h^2}{48L^2}$

f) Using the results of part e, we find

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = L\sqrt{\left(1 - \frac{2}{\pi^2}\right)} \quad \text{and} \quad \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{1}{4\sqrt{3}} \frac{h}{L}$$

g) Thus, $\Delta x \Delta p = \frac{h}{4\sqrt{3}}\sqrt{1 - \frac{2}{\pi^2}} = \sqrt{\frac{\pi^2 - 2}{3}}\left(\frac{\hbar}{2}\right) = 1.62\left(\frac{\hbar}{2}\right) \geq \frac{\hbar}{2}$ as required by the Heisenberg uncertainty principle.