Physics 133

Problem Set #1 Guide and Answers

Professor Mallinckrodt

Please note that (except where otherwise noted) these are decidedly *not* intended to model "good" solutions; you would need to add figures and *lots* of words of explanation. They are intended to help *you* work through problems that you may have had trouble with, *not* to demonstrate superior quality writeups. *Please* come see me if you have further questions. Thanks!

1.1 a) The two forces acting on the topmost charge have the same magnitude and form equal but opposite angles with respect to the $+\hat{x}$ direction. *Draw a picture* and finish the argument.

b) Again the two forces are equal with one in the $-\hat{x}$ direction and one that forms a 60 degree angle with the $-\hat{x}$ direction. Again *draw a picture* and finish the argument.

c) *Again* the two forces are equal. What are their magnitudes and directions? *Draw a picture*. Find the components of both forces and add them to find the components of the resultant. (Alternatively, simply show graphically that the resultant is the third side of an equilateral triangle.) The magnitude of the resultant force is $\frac{k_e q^2}{a^2}$.

d) There are six forces that must be vectorially added. Apply Newton's third law to each of the three *interaction pairs*. Use this to make a solid argument that the net force on the *system* is zero.

e) Likes repel whether they are positive or negative, unlikes attract. Consider what would happen if the sign of each charge was flipped.

f) 5.94 N

g) (The following *is* pretty much the answer! Just make sure you've thought about it and tried to answer it yourself before reading on.) (1) It works for *any* values of *q* and *a*. (2) It tells you exactly how the answer will *change* if you, for instance, double or triple *q* or *a*. (3) It is not subject to calculator error. (4) Despite all of these other advantages, it requires one less step!

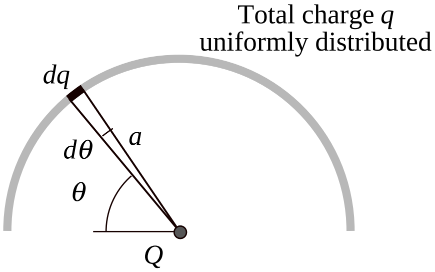
1.2 a) Use the hint. If *q*' is positive, *q* will force it to the left. *Q* will have to force it down and to the right which it cannot do. (Why not?) If *q*' is negative, *q* will force it to the right. *Q* will have to force it down and to the left (why?) which it *can* do, but *only* if *Q* is *also* negative. (Why?)

b) Find the *magnitudes* of the two forces on *q*' (in terms of *q*, *q*', and *Q*). Find the *x*-component of the force on *q*' due to *Q* which is $-\frac{\sqrt{3}}{2}\frac{k_e Q q'}{4a^2}$. (Be sure that you carefully consider the algebraic signs here. How do we *know* that this *is* a *negative* quantity?). Find the *x*-component of the force on *q*' due to *q* which is $-\frac{k_e q q'}{3a^2}$. (How do we know that *this* is a *positive* quantity?) Adding these two together gives the *x*-component of the *net* force on *q*' which is zero. Therefore, $Q = -\frac{8}{3\sqrt{3}}q$.

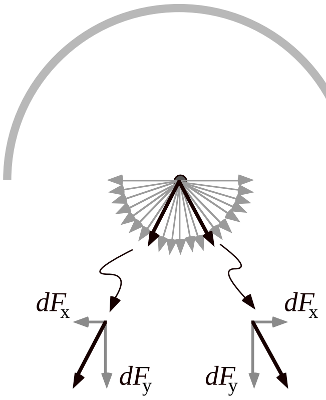
c) Find the *y*-component of the force on *q*' due to *Q*—which *is* the net force on *q*' (Why? Draw a picture.)—and substitute the result of part b to find $\vec{F} = -\frac{k_e Q q'}{8a^2}\hat{y} = \frac{k_e q q'}{3\sqrt{3}a^2}\hat{y}$. (Be sure that you see that *both* of these expressions indicate a vector in the $-\hat{y}$ direction.) Thus, $\frac{\vec{F}}{q'} = \frac{k_e q}{3\sqrt{3}a^2}\hat{y}$, a vector in the $+\hat{y}$ direction that doesn't depend on the value of q' as noted.

For the specific values given we get 784 kN/C.

1.3 (*This* solution is more fully written out and supported with figures in order to serve as an *example* of a good to excellent presentation.)



a) Since all of the charge q is a distance a away from the point charge Q as shown top right, it might seem that the force should have a magnitude $F = \frac{k_e Qq}{a^2}$. But force is a *vector* and the force on the point charge is the vector sum of an infinite number of infinitesimal contributions that are distributed over a range of directions spanning π radians as shown middle right. The result is a partial cancellation—in particular of the horizontal components—and this leads to a reduced net force.



b) Since both charges are positive, each infinitesimal vector contribution to the force on Q is *away* from the element dq of the semicircle that produces it. For each element on the left side of the semicircle (i.e., for $\theta < \pi/2$) there is a symmetrically located counterpart (at $\theta' = \pi - \theta$) that contributes a force of equal magnitude that has an equal but *opposite* horizontal (x) component as indicated in the middle and lower figures. As a result, *all* horizontal components cancel in pairs leaving the net force directly away from the midpoint of the semicircle.

c) The magnitude of the infinitesimal force from a given infinitesimal element at angle θ, is given by

$$dF = \frac{k_e Q\,dq}{a^2}.$$

The vertical (y) component of this force is

$$dF_y = dF \sin\theta .$$

Since the charge q is distributed *uniformly*, the infinitesimal charge dq is related to the infinitesimal range of angles $d\theta$ on which it resides by a proportionality

$$\frac{dq}{q} = \frac{d\theta}{\pi} \quad \Rightarrow \quad dq = \frac{q\,d\theta}{\pi}$$

The magnitude of the net force is obtained by adding up all of the *vertical components* of the infinitesimal contributions. Thus,

$$F = \int dF_y = \int dF \sin\theta = \int \frac{k_e Q dq}{a^2} \sin\theta = \int \frac{k_e Q q d\theta}{\pi a^2} \sin\theta = \frac{k_e Q q}{\pi a^2} \int_0^\pi \sin\theta\, d\theta = \frac{2}{\pi}\frac{k_e Qq}{a^2} \quad \text{or} \quad \vec{F} = \frac{2}{\pi}\frac{k_e Qq}{a^2}, \text{“}down\text{”}.$$

d) Clearly this answer is smaller than the “naïve” answer $\frac{k_e Qq}{a^2}$ by a factor of $\frac{2}{\pi} = 0.637$.

1.4 a) All individual forces are increased in magnitude by a factor of two without changing their directions. Draw a vector diagram showing what happens to the sum. Result: $\vec{F} = 10.46$ mN, in the same direction.

b) This would “flip” the direction of each individual force. So what happens to the resultant? Again, draw a vector diagram. Result: $\vec{F} = 5.23$ mN, in the opposite direction.

c) By the same logic, we simply change the force by the same factor (20,670). (Why? Explain or draw a vector diagram.) $\vec{F} = 108$ N, in the same direction.

d) Do the reverse. The *force* is changed by a factor of -8.795 and so is the charge. $q = -10.6\ \mu$C.

e) No. (Why not? Try drawing some possible vector diagrams for different possible charges.)

f) $\vec{E} = \frac{\vec{F}}{q} = \frac{\text{5.23 mN, in the direction shown}}{+1.20\ \mu\text{C}}$ = 4.36 kN/C, in the direction shown

g) No. You’d need to know either the location and magnitude of all charges *or* the force on some charge at a specific new location of interest.

1.5 a) Keep in mind that the +q creates a contribution to the net E-field that is *away* from it and the -2q creates a contribution to the net E-field that is *toward* it. We need to consider three regions separately. (Draw pictures of the contributions for each one!)

For $x < 0$:

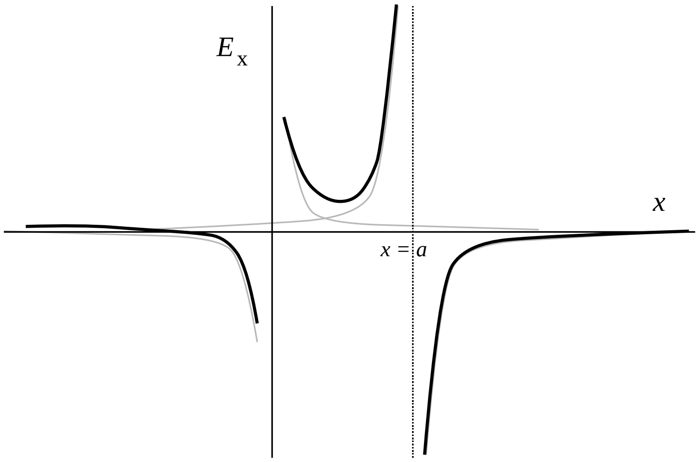
$$E_x = -\frac{k_e|q|}{|x|^2} + \frac{k_e|-2q|}{(|x|+a)^2} = -\frac{k_e q}{x^2} + \frac{2k_e q}{(x-a)^2}$$

For $0 < x < a$

$$E_x = +\frac{k_e|q|}{x^2} + \frac{k_e|-2q|}{(a-x)^2} = \frac{k_e q}{x^2} + \frac{2k_e q}{(x-a)^2}$$

For $a < x$

$$E_x = +\frac{k_e|q|}{x^2} - \frac{k_e|-2q|}{(x-a)^2} = \frac{k_e q}{x^2} - \frac{2k_e q}{(x-a)^2}$$



A graph (not a "map") is sketched at right showing the individual contributions and the net x-component. Note that the field is always to the right between the charges, always to the left for $x > a$. For $x < 0$, the field is to the left *near* the positive charge at the origin, but *far* from the origin the field from the farther but larger negative charge at $x = a$ dominates and results in a field to the right. Therefore, *somewhere* along the negative x axis, the two contributions must cancel! Somewhere a little farther along to the left the field will have a local maximum before it starts to monotonically *decrease* in magnitude.

b) We already know this happens for $x < 0$. Let's just find the distance by setting the magnitudes equal.

$$\frac{k_e q}{|x|^2} = \frac{2k_e q}{(|x|+a)^2} \quad \Rightarrow \quad x = -\frac{a}{2^{1/2}-1} = -2.414\ a$$

1.6 a) This is the same physical situation as in Problem 5 and we have already seen that the field *does* have a local minimum between the two charges. Differentiate wrt x and set the result equal to zero. Find

$$\frac{2}{x^3} + \frac{4}{(x-a)^3} = 0 \quad \text{or} \quad x = \frac{a}{2^{1/3}+1} = 0.442\ a$$

b) We have also seen (although it may not be *quite* as obvious!) that the field has a local maximum in the region $x < a$. Differentiate wrt x and setting the result equal to zero. Find

$$-\frac{2}{x^3} + \frac{4}{(x-a)^3} = 0 \quad \text{or} \quad x = \frac{a}{1-2^{1/3}} = -3.85\ a$$

1.7 Review the hints and then show—with a figure *and* an explanation—why each infinitesimal piece of the rod produces an infinitesimal electric field at the point P having a magnitude $dE = \dfrac{k\,dq}{a^2+s^2}$ where $dq = \dfrac{Q}{a}ds$ and Q is the total charge on the rod. Now, using your figure and some appropriate trigonometry, explain why the rightward (x) and downward (y) components of $\vec{E}$ are given by

$$dE_x = dE\frac{s}{(a^2+s^2)^{1/2}} \quad \text{and} \quad dE_y = dE\frac{a}{(a^2+s^2)^{1/2}}$$

Next, explain why you get the following integrals.

$$E_x = \frac{kQ}{a}\int_0^a \frac{s\,ds}{(a^2+s^2)^{3/2}} = \left(1-\sqrt{2}/2\right)\frac{kQ}{a^2}$$

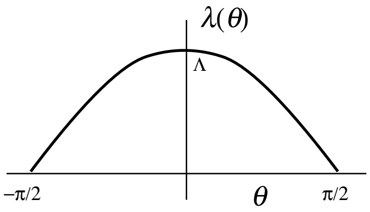
$$E_y = kQ\int_0^a \frac{ds}{(a^2+s^2)^{3/2}} = \sqrt{2}/2\,\frac{kQ}{a^2}$$

Use your results to explain why the direction of $\vec{E}$ is $\tan^{-1}\left(\dfrac{1}{\sqrt{2}-1}\right) = \dfrac{3}{8}\pi =$ 67.5° clockwise from "to the right"

1.8 a) The charge is distributed preferentially near the *midpoint* of the semicircle.

b) The sketch is at right.

c) The symmetry argument is the same as that given in problem 3.



d) Add up the infinitesimal charges on all of the infinitesimal pieces. Each such is $dq = \lambda ds = (\Lambda\cos\theta)(ad\theta)$ so $q = \int dq = \Lambda a\int_{-\pi/2}^{\pi/2}\cos\theta d\theta = 2\Lambda a$

e) We follow the same procedure as was used in problem 3. The only differences are that we have defined our angle θ differently and $dq = \Lambda a\cos\theta d\theta$. Thus,

$$F = \int dF_{\mathrm{v}} = \int dF\cos\theta = \int\frac{k_eQdq}{a^2}\cos\theta = \int\frac{k_eQ\Lambda a}{a^2}\cos^2\theta d\theta = \frac{k_eQ\Lambda a}{a^2}\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta = \frac{\pi}{2}\frac{k_eQ\Lambda a}{a^2}$$

f) Using the result of part d to eliminate Λ, we find $F = \frac{\pi}{4}\frac{k_eQq}{a^2}$.

g) Clearly, and as expected, the answer is (as in problem 3) *smaller* than $\frac{k_eQq}{a^2}$.

h) On the other hand, this answer *is* larger than that for problem 3 ($\frac{2}{\pi} = 0.637 < \frac{\pi}{4} = 0.785$) as expected since the charge is now distributed in a more localized fashion producing *less* cancellation.

1.9 a) Each individual charge experiences three forces: A *repulsive* force (from the charge diametrically opposite) and *two* attractive forces (from the other two charges) of equal *magnitudes*. The two attractive forces produce a net force *toward* the center of mass of the configuration that is given by $2\left(\frac{k_eq^2}{2a^2}\right)\frac{\sqrt{2}}{2} = \frac{\sqrt{2}k_eq^2}{2a^2}$. (Draw a picture, add the two vectors by components, and see—very easily—why this is true.) Thus the *net* force produced by all three charges is just

$$\vec{F} = \frac{k_eq^2}{a^2}\left(\frac{1}{4},\ \text{away from the CM} + \frac{\sqrt{2}}{2},\ \text{toward the CM}\right) = \ (\textit{You}\ \text{simplify the result})$$

b) As long as the square configuration is maintained, the force on each charge will be constant and toward the common CM. This is precisely the condition needed for uniform circular motion. We need to give each charge a shove tangent to a circle drawn through all four of them giving each of them a speed *v* such that Newton's second law is obeyed! That is $F = m\frac{v^2}{a}$. (*You* solve to find the required "just right" value of *v*!) The square configuration of charges will then simply rotate around the CM.

c) (Remember that the *givens* are *m*, *a*, and *q*!) The period is the time it takes each charge to complete one full trip around the circumference of the circle and is given by

$$T = \frac{2\pi a}{v} = (\textit{you}\ \text{do the algebra to show}) = \frac{4\pi a}{q}\sqrt{\frac{ma}{\left(2\sqrt{2}-1\right)k_e}}$$

d) Use the fact that the dimensions of k_e are the same as $\frac{Fr^2}{q^2}$, i.e., $\frac{\text{force}\cdot\text{length}^2}{\text{charge}^2} = \frac{\text{mass}\cdot\text{length}^3}{\text{time}^2\cdot\text{charge}^2}$ and the formula from part c to show that the result has dimensions of time.